WHEN IS A DERIVATOR REPRESENTABLE?

G. RAPTIS

Let $D$ be a prederivator. The underlying diagram functors define a morphism of prederivators

$$\text{dia}_D : D \to \mathcal{C}at(-, D(e)).$$

A prederivator $D$ is called representable if $\text{dia}_D$ is an equivalence of prederivators.

Let $\mathcal{C}$ be a right derivable category in the sense of [1]. The associated prederivator $D(\mathcal{C})$ is a right derivator satisfying (Der5) [1, 2.21]. Suppose that the representable prederivator

$$X \mapsto \mathcal{C}at(X, \text{Ho}(\mathcal{C}))$$

is also a right derivator and that the morphism $\text{dia}_{D(\mathcal{C})}$ is cocontinuous. Then we may regard the homotopy category $\text{Ho}(\mathcal{C})$ as a right derivable category where every morphism is a cofibration and the weak equivalences are the isomorphisms. Moreover, the canonical functor

$$\gamma : \mathcal{C} \to \text{Ho}(\mathcal{C})$$

is right exact in the sense of [1, 1.9]. Then $\text{dia}$ is simply the morphism induced by $\gamma$. Since $\text{dia}(e)$ is obviously an equivalence, [1, 3.20] implies that $\text{dia}$ is also an equivalence of right derivators, i.e., $D(\mathcal{C})$ is representable.

The fact that $\text{Ho}(\mathcal{C})$ admits (co)limits is by itself insufficient for the argument. A counterexample is the derivator associated to the Waldhausen category of finitely generated stable modules over $\mathbb{Z}/p^2$. The homotopy category is equivalent to the category of finite dimensional $F_p$-vector spaces (see [2, p. 1831]). Thus it is essential to know that $\text{dia}$ is also cocontinuous (cf. [2, 4.5]).

On the other hand, assuming that $\text{dia}$ is cocontinuous, it does not follow that $\gamma$ is an equivalence. A trivial counterexample is the right derivable category of sets where every map is a weak equivalence. Its homotopy category is the terminal category.

It remains open whether $D$ is always representable if $\text{dia}_D$ is cocontinuous. If this fails in general, it would show an interesting feature of derivators with (good) models. On the other hand, showing that this is true seems to require an extension of [1, 3.19, 3.20] to abstract derivators.

References


E-mail address: georgios.raptis@mathematik.uni-regensburg.de

Date: June 30, 2014.