
Seminar: Introduction to K -theory
(every Thursday, 10 am – 1 pm, M2)

Description. This seminar will be an introduction to algebraic K -theory, with an emphasis on its basic ideas, examples, and connections with other areas of mathematics. We will begin with group completion of an abelian monoid and the theory of projective modules, which together lead to the definition of K_0 of a ring (Session 1). Then we will study some of the fundamental properties of K_0 , as well as examples arising from important classes of rings such as local rings and Dedekind domains (Sessions 2-4). In addition, we will discuss the example of vector bundles and the connection with topological K -theory (Session 5). We will also see in detail how K_0 can be understood from a categorical point of view and examine the benefits of this perspective (Sessions 6-8).

After reviewing the general linear group and elementary matrices, we will then introduce K_1 of a ring, study some of its fundamental properties, and discuss examples of computations (Sessions 9-10). We will conclude with an introduction to K_2 of a ring and, if time permits, a glimpse of higher algebraic K -theory (Sessions 11-12). Along the way, we will encounter connections with (homological) algebra, number theory, (algebraic) topology and geometry.

The seminar should be of interest to anyone interested in algebra, homological algebra, algebraic topology, or the connections between algebra and topology.

References. The main references for the seminar are the books by Milnor [4], Rosenberg [5] and Weibel [6]. Other useful references include: Bass [1] (a classic), Hatcher [2] and Karoubi [3] (for topological K -theory).

Prerequisites. Basic theory of rings and modules. Basic notions from topology (topological spaces, compactness, vector bundles, etc.) and category theory (categories, functors, natural transformations, etc.).

Format and Assessment. The seminar consists of twelve main sessions, divided into three parts. Each session contains three 45-minute talks. The first two talks develop the mathematical core of each session, while the third talk, given by the lecturer, is supplementary and will be devoted to examples, motivation and overview, as well as applications.

Each participant will give **two 45-minute talks** (in Greek or English), one from each of two different sessions belonging to two different parts; these talks will be chosen from among the first two talks of the respective sessions. The speakers will have the opportunity to discuss the preparation of their talks with the lecturer. In addition, participants are required to submit a **written essay and/or report** for each of their talks, whose exact topic will be determined in consultation with the lecturer. The participants' grade for the seminar will be based on the two presentations and the corresponding two essays.

Preliminary Meeting. There will be a preliminary meeting (Session 0) on **01/10/2026** (10 am - 12 pm, M2) to discuss the program of the seminar and distribute the talks.

Registration. Please e-mail raptisg@math.auth.gr in order to express your interest, register for the seminar and, if you wish, indicate any preferences regarding the talks you would like to give.

Session 0 (01/10): Preliminary Meeting

Note: The program below is provisional and may be adjusted depending on the background and interests of the participants.

PART 1: K_0 OF RINGS

Session 1 (08/10)

Talk 1.1: *The group completion of an abelian monoid.* Abelian monoids. Group completion. Universal property of group completion. Explicit description of group completion. Cancellation property. Stable equivalence. Semirings and group completion. Examples. Main references: [6, pp. 69-73 and Exercises], [5, 1.1].

Talk 1.2: *Projective modules.* Free modules. IBP and examples. Stably free modules and examples. Projective modules. Characterization of projective modules. Examples. Main references: [6, pp. 1-4 and Exercises, pp. 8-10], [5, 1.1]. See also [1] or any textbook on module theory.

Talk 1.3: *Comments on IBP and examples. Projective modules via idempotents.*

Session 2 (15/10)

Talk 2.1: *K_0 of a ring.* Definition of $K_0(R)$. Functoriality of K_0 . The Eilenberg swindle. Ring structure. K_0 and products of rings. Examples: K_0 of flasque rings, fields and PIDs (assuming the structure theorem for finitely generated modules). Main references: [6, pp. 74-75], [5, 1.1, 1.3, pp. 1-7 and 11-12], [4, §1, pp. 3-8].

Talk 2.2: *Local rings.* $\tilde{K}_0$. Local rings. Projective modules over a local ring. K_0 of a local ring. Examples. Main reference: [5, 1.3, pp. 12-16], [4, §1, pp. 3-8]. See also [6, pp. 76-78], [1].

Talk 2.3: *K_0 via idempotents and applications. Morita invariance.*

Session 3 (22/10)

Talk 3.1: *Milnor patching.* Milnor squares. Descent data and Milnor patching. Main references: [6, pp. 14-16 and Exercises], [4, §2].

Talk 3.2: *Mayer-Vietoris sequence for K_0 .* Mayer-Vietoris sequence associated to a Milnor square. Examples. Main references: [6, pp. 83-85 and Exercises], [4, §2-3]. See also [1], [5, 1.5].

Talk 3.3: *Relative K_0 and excision.*

Session 4 (29/10)

Talk 4.1: *Dedekind domains.* Dedekind domains. Fractional ideals. Class group. Projective modules over Dedekind domains. Main references: [5, pp. 16-19], [4, §1, pp. 9-15].

Talk 4.2: *K_0 of Dedekind domains.* Classification of finitely generated projective modules over a Dedekind domain. K_0 of a Dedekind domain. Characterization of Dedekind domains. Main references: [5, pp. 19-22], [4, §1, pp. 9-15].

Talk 4.3: *Examples of Dedekind domains and connections with number theory.*

Session 5 (05/11)

Talk 5.1: *Topological K-theory.* Review of vector bundles and operations on vector bundles. $K_0(X)$. Pullback and functoriality. Swan's theorem. Main reference: [5, pp. 32-36]. See also [2, Chapter 1 and 2.1], [3, Ch. I-II].

Talk 5.2: *Properties of topological K-theory.* Exact sequences in topological K -theory. Homotopy invariance of topological K -theory. Main reference: [5, pp. 36-40]. See also [2, 2.1-2.2], [3, Ch. I-II].

Talk 5.3: *Classification of vector bundles and topological K-theory as a generalized cohomology theory.*

PART 2: K_0 OF CATEGORIES**Session 6** (12/11)

Talk 6.1: K_0 of an abelian category. Abelian categories. Examples. Morita equivalence. K_0 . Functoriality. Examples. The Cartan homomorphism. Main reference: [6, pp. 124-127 and Exercises, pp. 82-83]. See also [5, 3.1].

Talk 6.2: *Devissage and Localization.* The devissage theorem for K_0 . Examples. The localization theorem for K_0 . Applications. Main reference: [6, pp. 129-132 and Exercises]. See also [5, 3.1].

Talk 6.3: *The Fundamental Theorem for G_0 . Calculations and applications.*

Session 7 (19/11)

Talk 7.1: K_0 of an exact category. Exact categories. Examples. K_0 . Functoriality. Examples. Cofinality. Morita invariance (revisited). Main references: [6, pp. 140-144 and Exercises], [5, 3.1].

Talk 7.2: *Resolution and $K_0(R[t])$.* Euler characteristics. The resolution theorem for K_0 . Examples. The Fundamental Theorem for K_0 of regular rings. Main references: [6, pp. 146-150 and Exercises], [5, 3.1-3.2].

Talk 7.3: *The Fundamental Theorem for K_0 and negative K-theory.*

Session 8 (26/11)

Talk 8.1: K_0 of Waldhausen categories. Waldhausen categories. Examples. K_0 . Functoriality. Examples. The Localization Theorem. Main reference: [6, pp. 172-175, 181-182].

Talk 8.2: *Bounded chain complexes.* Chain complexes as a Waldhausen category. The Gillet-Waldhausen Theorem for K_0 . Main reference: [6, pp. 175-180].

Talk 8.3: *An Application: Wall's Finiteness Obstruction*

PART 3: K_1 AND K_2 OF RINGS**Session 9** (03/12)

Talk 9.1: K_1 of a ring. General linear groups and elementary matrices. Whitehead's lemma. Definition of $K_1(R)$. Functoriality. Examples. Main references: [6, pp. 197-201 and Exercises], [5, 2.1], [4, §3].

Talk 9.2: *Examples of K_1 .* K_1 of fields/division rings, local rings and Euclidean domains. Morita invariance for K_1 . Main reference: [5, pp. 62-69, 74-75]. See also [6, III.1], [4, §3].

Talk 9.3: *Alternative approaches to K_1 and applications.*

Session 10 (10/12)

Talk 10.1: *Relative K_1 .* Milnor patching for K_1 . The Mayer-Vietoris sequence associated to a Milnor square. $K_1(R, I)$. Examples. Main references: [6, III.2 and Exercises], [5, 2.5], [4, §4].

Talk 10.2: *The Fundamental Theorem for G_1 .* An overview of the proof of the Fundamental Theorem for G_1 and for K_1 of regular rings. Main references: [6, III.3], [5, 3.2].

Talk 10.3: *Another application: Whitehead torsion and its uses.*

Session 11 (17/12)

Talk 11.1: *K_2 of a ring.* The Steinberg group. $K_2(R)$. Universal central extensions (sketch). Functoriality. Morita invariance. Main references: [6, pp. 236-241], [5, 4.2], [4, §5].

Talk 11.2: *Computations of K_2 .* Possible topics (some details will be omitted): (i) K_2 -classes from K_1 . (ii) Steinberg symbols. Matsumoto's theorem (without proof) and consequences. (iii) Connections with number theory. (iv) $K_2(\mathbb{Z})$. (v) Localization and K_2 of Dedekind domains. Main references (selection): [6, pp. 245-248, 251-262], [5, 4.2, 4.4], [4, §8-11, §13].

Talk 11.3: *Computations of K_2 (ctd.)*

Session 12 (tbd)

Talk 12.1: *Computations of K_2 (ctd.)*

Talk 12.2: *Computations of K_2 (ctd.)*

Talk 12.3: *Towards Higher Algebraic K -theory.*

REFERENCES

- [1] H. Bass, *Algebraic K-Theory*, Mathematics Lecture Notes Series, W. A. Benjamin, New York–Amsterdam, 1968.
- [2] A. Hatcher, *Vector Bundles and K-Theory*, 2003, available at <https://pi.math.cornell.edu/~hatcher/VBKT/VBpage.html>.
- [3] M. Karoubi, *K-Theory: An Introduction*, Grundlehren der mathematischen Wissenschaften, vol. 226, Springer-Verlag, Berlin–New York, 1978.
- [4] J. Milnor, *Introduction to Algebraic K-Theory*, Annals of Mathematics Studies, no. 72, Princeton University Press, Princeton, NJ, 1971.
- [5] J. Rosenberg, *Algebraic K-Theory and Its Applications*, Graduate Texts in Mathematics, vol. 147, Springer-Verlag, New York, 1994.
- [6] C. A. Weibel, *The K-book: An Introduction to Algebraic K-Theory*, Graduate Studies in Mathematics, vol. 145, American Mathematical Society, Providence, RI, 2013.