

Seminar

Catastrophe Theory/Katastrophentheorie

(Singularitäten differenzierbarer Abbildungen)

Germes, Jets, Determinacy, Codimension

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Theorem (Lemma of Borel)

The map $T: \mathcal{E}_n \rightarrow \mathbb{R}[[x_1, \dots, x_n]]$ defined by $T[f] = \sum_{\nu \in \mathbb{N}_0^n} \frac{1}{\nu!} D^\nu f(\mathbf{0}) x^\nu$ is a surjective algebra homomorphism.

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where $u(x)$ is a smooth function such that $0 \leq u(x) \leq 1$, $u(x) = 1$ if $|x| \leq \frac{1}{2}$, and $u(x) = 0$ if $|x| \geq 1$.

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$$\left| \frac{d^k}{dx^k} \left(\frac{1}{n!} 2^{n^2} u(2^{n^2+1}x) x^n \right) \right| \leq A_{k,u} \cdot 2^{-n}$$

holds independently of x .

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The qualitative properties of a germ should be invariant under (local) changes of coordinates. Thus, we restrict our attention to understanding the various types of germs in $\mathcal{E}_n$ up to equivalence.

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Corollary

Non-degenerate critical points are isolated (as critical points).

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- Assume $f(\mathbf{0}) = 0$. By precomposing with a linear isomorphism $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ if necessary, we may also assume that the symmetric matrix $D^2f(\mathbf{0})$ is the diagonal matrix $2I_{n,r,s}$ where $I_{n,r,s} = (-I_s) \oplus I_{r-s} \oplus 0_{n-r}$.

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Then it suffices to show that $[k]$ is equivalent to $[q_{r,s} \circ p_{\leq r}]$ since

$$(x, y) \mapsto (x + h(y), y) \text{ is a local diffeomorphism.}$$

The Reduction Lemma: Sketch of the Proof. II

Theorem (The Reduction Lemma)

Let $[f] \in \mathcal{E}_n$ be a degenerate critical germ of rank $r < n$. Then $[f]$ is reducible to a totally degenerate critical germ $[g] \in \mathcal{E}_{n-r}$.

- (ctd.) We have: (1) $k(\mathbf{0}, y) = 0$, (2) $(D_1 k)(\mathbf{0}, y) = \mathbf{0}$, and (3) $(D_1^2 k)(\mathbf{0}, \mathbf{0}) = 2I_{r,s}$.

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$$K(x, y) = (K_{i,j}(x, y))$$

such that $K(\mathbf{0}, \mathbf{0}) = I_{r,s}$ and $k(x, y) = \langle x, K(x, y)x \rangle$.

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- As in the proof of the Morse Lemma, we can find an $(r \times r)$ -matrix of smooth maps

$$T(x, y) = (T_{i,j}(x, y))$$

such that $T(\mathbf{0}, \mathbf{0}) = I_r$ and

$$K(x, y) = T(x, y)^t K(\mathbf{0}, \mathbf{0}) T(x, y).$$

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where $\phi(x, y) = (T(x, y)x, y)$ is a local diffeomorphism since

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Theorem (Uniqueness of the Residual Singularity)

Let $[f]$ and $[g]$ be totally degenerate critical germs in $\mathcal{E}_k$ with $f(\mathbf{0}) = g(\mathbf{0}) = 0$. Suppose that the following germs in $\mathcal{E}_{r+k}$

$$[q_{r,s} \circ p_{\leq r}] + [f \circ p_{> r}] \text{ and } [q_{r,s} \circ p_{\leq r}] + [g \circ p_{> r}]$$

are equivalent. Then $[f] \stackrel{\text{equiv}}{\sim} [g]$ in $\mathcal{E}_k$.

k -equivalence and Jets

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Let k be a positive integer. Two germs $[f], [g] \in \mathcal{E}_n$ are **k -equivalent** if

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The class of germs k -equivalent to $[f]$ is the k -**jet** of $[f]$, denoted $j^k[f]$.

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The set of k -jets of germs in $\mathcal{E}_n$ is an $\mathbb{R}$ -vector space J_n^k and the map

$$J_n^k \rightarrow P_n^k: = \{p \in \mathbb{R}[x_1, \dots, x_n] \mid \deg(p) \leq k\}, [f] \mapsto T_f^k$$

is a linear isomorphism. Thus, $\dim_{\mathbb{R}} J_n^k = \binom{n+k}{k}$.

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The middle equality is a consequence of Taylor's theorem.

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Examples. Two non-critical germs $[f]$ and $[g]$ may have very different Taylor polynomials, but they are always equivalent. On the other hand, a non-trivial germ in m_n^{k+1} is k -equivalent to $[0]$, but it is not equivalent to $[0]$.

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- 1 A germ $[f] \in m_1$ has determinacy k if and only if k is the smallest positive integer such that $f^{(k)}(0) \neq 0$.

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All of these implications are strict.

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$$[f] = [h_0(x, 0)] \stackrel{\text{equiv}}{\sim} [h_s(x, 0)] \stackrel{\text{equiv}}{\sim} [h_1(x, 0)] = [g], \text{ as required.}$$

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which implies $[g] \in \langle m_n \mathcal{J}[f] \rangle$ as required. It turns out that the existence of such a smooth Φ can be shown at the (finite dimensional) level of k -jets, which suffices for the proof.

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(4): $\mathbf{0}$ is not an isolated critical point in this case and the result follows from (3).

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The following result follows easily from these considerations.

Theorem

Let $[f] \in m_n^2$. Then $\text{cod}[f] < \infty$ if and only if $\det[f] < \infty$. Moreover,

$$\det[f] - 2 \leq \text{cod}[f] \leq \binom{n + \det[f]}{n} - 1.$$

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$$\mathcal{J}([q_{r,s} \circ p_{\leq r}] + [g \circ p_{> r}]) = \langle \mathcal{J}[q_{r,s}] \rangle_{\mathcal{E}_n} + \langle \mathcal{J}[g] \rangle_{\mathcal{E}_n} = \langle m_r \rangle_{\mathcal{E}_n} + \mathcal{J}[g].$$

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Let $[f] \in m_n^2$ of rank $r \leq n$. The **corank** of $[f]$ is $(n - r)$ (i.e., the dimension of the kernel of the Hessian matrix $D^2f(\mathbf{0})$) and is denoted by $\text{cor}[f]$.

Theorem

Let $[f] \in m_n^2$. Then:

- 1 Suppose that $[f]$ has positive corank and reduces to $[g] \in m_{n-r}^3$. Then $\text{cod}[f] = \text{cod}[g]$.
- 2 Suppose that $\text{cod}[f] < \infty$. Then (as $\mathbb{R}$ -vector spaces)

$$\mathcal{J}[f] = \langle m_n \mathcal{J}[f] \rangle_{\mathcal{E}_n} \oplus \mathbb{R}\{D_1 f\} \oplus \cdots \oplus \mathbb{R}\{D_n f\}.$$

$$\text{Therefore } \text{cod}[f] = (\dim_{\mathbb{R}} m_n / \langle m_n \mathcal{J}[f] \rangle) - n = \dim_{\mathbb{R}} m_n^2 / \langle m_n \mathcal{J}[f] \rangle.$$

- 3 $\text{cod}[f] \geq \frac{k(k+1)}{2}$ where $k = \text{cor}[f]$.

Sketch of the Proof. (1) Suppose $[f] \stackrel{\text{equiv}}{\sim} [q_{r,s} \circ p_{\leq r}] + [g \circ p_{> r}]$. The result follows from the following identification of $\mathbb{R}$ -vector subspaces:

$$\mathcal{J}([q_{r,s} \circ p_{\leq r}] + [g \circ p_{> r}]) = \langle \mathcal{J}[q_{r,s}] \rangle_{\mathcal{E}_n} + \langle \mathcal{J}[g] \rangle_{\mathcal{E}_n} = \langle m_r \rangle_{\mathcal{E}_n} + \mathcal{J}[g].$$

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(2) An element in $\mathcal{J}[f]$ can be easily written as a sum of elements from the given vector subspaces. It is non-trivial to prove that the sum is direct (this uses that the codimension is finite!).

Codimension and Corank (ctd.)

Let $[f] \in m_n^2$ of rank $r \leq n$. The **corank of $[f]$** is $(n - r)$ (i.e., the dimension of the kernel of the Hessian matrix $D^2f(\mathbf{0})$) and is denoted by $\text{cor}[f]$.

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Theorem

Let $[f] \in m_n^2$. Then:

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② Suppose that $\text{cod}[f] < \infty$. Then (as $\mathbb{R}$ -vector spaces)

$$\mathcal{J}[f] = \langle m_n \mathcal{J}[f] \rangle_{\mathcal{E}_n} \oplus \mathbb{R}[D_1 f] \oplus \cdots \oplus \mathbb{R}[D_n f].$$

$$\text{Therefore } \text{cod}[f] = (\dim_{\mathbb{R}} m_n / \langle m_n \mathcal{J}[f] \rangle) - n = \dim_{\mathbb{R}} m_n^2 / \langle m_n \mathcal{J}[f] \rangle.$$

③ $\text{cod}[f] \geq \frac{k(k+1)}{2}$ where $k = \text{cor}[f]$.

Sketch of the Proof. (3): $\text{cod}[f] = \text{cod}[g] = \dim_{\mathbb{R}} \mathcal{E}_k / \langle m_k \mathcal{J}[g] \rangle - k - 1$ (using (2)) and $\dim_{\mathbb{R}} \mathcal{E}_k / \langle m_k \mathcal{J}[g] \rangle \geq \dim_{\mathbb{R}} \mathcal{E}_k / m_k^3 = \binom{k+2}{2}$. □

Thom's Classification of Germs of Codimension ≤ 4

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The following theorem classifies all degenerate critical germs $[f]$ with $1 \leq \text{cod}[f] \leq 4$. Note that the possibilities for the corank are as follows:

- $\text{cod}[f] = 1$ or $2 \Rightarrow \text{cor}[f] = 1$.
- $\text{cod}[f] = 3$ or $4 \Rightarrow \text{cor}[f] = 1$ or 2 .

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Theorem

Let $[f] \in m_n^2$.

- ① Suppose that $\text{cor}[f] = 1$ and $1 \leq \text{cod}[f] \leq 4$. Then the residual singularity of $[f]$ is equivalent to one of the following germs:

$[x^3], [x^4], [-x^4], [x^5], [x^6], [-x^6]$ (4 possibilities up to sign).

- ② Suppose that $\text{cor}[f] = 2$ and $\text{cod}[f] = 3$. Then the residual singularity of $[f]$ is equivalent to one of the following germs:

$[x^3 - xy^2], [x^3 + y^3]$ (2 possibilities).

- ③ Suppose that $\text{cor}[f] = 2$ and $\text{cod}[f] = 4$. Then the residual singularity of $[f]$ is equivalent to one of the following germs:

$[x^2y + y^4], [-x^2y - y^4]$ (1 possibility up to sign).

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