

Seminar

Catastrophe Theory/Katastrophentheorie

(Singularitäten differenzierbarer Abbildungen)

Classification of Germs with Codimension ≤ 4

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Introduction

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- 2 If $[f] \in \mathcal{m}_n^2$ (i.e., $[f] \in \mathcal{m}_n$ and $[f]$ is critical) and $\text{cod}[f] = 0$, then $[f]$ is a non-degenerate critical germ.

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- 2 If $[f] \in m_n^2$ (i.e., $[f] \in m_n$ and $[f]$ is critical) and $\text{cod}[f] = 0$, then $[f]$ is a non-degenerate critical germ. The Morse Lemma shows that $[f]$ is equivalent to the germ of the function

$$(x_1, \dots, x_n) \mapsto -x_1^2 - \dots - x_s^2 + x_{s+1}^2 + \dots + x_n^2$$

where s is the index of $[f]$ (i.e., the index of the Hessian matrix $D^2f(\mathbf{0})$).

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Goal (Today): Classify degenerate critical germs $[f] \in \mathcal{m}_n^2$ with $1 \leq \text{cod}[f] \leq 4$ (up to equivalence).

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Assuming a classification of the respective totally degenerate germs in $\mathcal{M}_{n-r}^3$, then the germ $[f] \in \mathcal{M}_n^2$ is equivalent to exactly one of these totally degenerate germs up to the addition of the non-degenerate normal quadratic form of index s in the **remaining r variables**:

$$-x_1^2 - \cdots - x_s^2 + x_{s+1}^2 + \cdots x_r^2$$

where s is the index of $[f]$.

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We will treat each one of these cases separately.

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In other words, $\text{cor}[f] = 1$ and $1 \leq \text{cod}[f] \leq 4$ if and only if $[f]$ is equivalent to the germ of the function:

$$(x_1, \dots, x_n) \mapsto -x_1^2 - \dots - x_s^2 + x_{s+1}^2 + \dots + x_{n-1}^2 + g(x_n)$$

where $0 \leq s \leq n-1$ is the index of $[f]$ and $g(x)$ is (exactly) one of the polynomials listed in the theorem above.

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$$\det[g] \in \{3, 4, 5, 6\}.$$

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Claim: The germ of $g: \mathbb{R}^2 \rightarrow \mathbb{R}, (x, y) \mapsto x^3 - xy^2$ has codimension = 3.

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Proof of the Claim. We need to identify a minimal set of germs such that together with $\mathcal{J}[g] = \langle 3x^2 - y^2, xy \rangle_{\mathcal{E}_n} \subseteq \mathfrak{m}_2$ span the vector space $\mathfrak{m}_2$.

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So $\text{cod}[g] \leq 3$.

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Proof of the Claim. The germs $[x], [y], [xy]$ define a basis for $m_2/\mathcal{I}[g]$. The proof is similar (and easier). □

Case B: Proof of the Theorem (ctd.)

(1) $\Rightarrow$ (2): We need to classify germs $[g] \in \mathfrak{m}_2^3$ with codimension $\text{cod}[g] = 3$ up to equivalence. Note that $\text{cor}[g] = 2$ in this case (because the germ $[g]$ is totally degenerate).

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For this purpose, we will use the direct sum decomposition:

$$\mathcal{J}[g] = \langle m_2 \mathcal{J}[g] \rangle_{\mathcal{E}_2} \oplus \mathbb{R} \left\{ \frac{\partial g}{\partial x} \right\} \oplus \mathbb{R} \left\{ \frac{\partial g}{\partial y} \right\}.$$

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(We will actually only need the – much easier – fact that the vector subspaces $\langle m_2 \mathcal{J}[g] \rangle_{\mathcal{E}_2}$, $\mathbb{R} \left\{ \frac{\partial g}{\partial x} \right\}$, and $\mathbb{R} \left\{ \frac{\partial g}{\partial y} \right\}$ span $\mathcal{J}[g]$.)

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Therefore $m_2^3 = \langle m_2 \mathcal{J}[g] \rangle_{\mathcal{E}_2}$ and, in particular, $[g]$ is 3-complete.

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The last Claim implies that $[g] \stackrel{\text{equiv}}{\sim} [T_g^3]$ where $\det[T_g^3] = 3$.

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Note that since $[g] \in m_2^3$ (i.e., $[g]$ is critical and totally degenerate), T_g^3 is cubic form on $\mathbb{R}^2$.

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Classification of Cubic Forms on $\mathbb{R}^2$: Let $p(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$ be a cubic form on $\mathbb{R}^2$. There is a linear coordinate transformation ϕ on $\mathbb{R}^2$ such that $p \circ \phi$ is exactly one of the following cubic forms on $\mathbb{R}^2$:

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Claim.

(a) $\text{cod}[0] = \text{cod}[x^3] = \text{cod}[x^2y] = \infty.$

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Proof of the Claim.

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Classification of Cubic Forms on $\mathbb{R}^2$: Let $p(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$ be a cubic form on $\mathbb{R}^2$. There is a linear coordinate transformation ϕ on $\mathbb{R}^2$ such that $p \circ \phi$ is exactly one of the following cubic forms on $\mathbb{R}^2$:

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Thus, $[T_g^3] (\overset{\text{equiv}}{\sim} [g])$ is equivalent to one of the following two germs:

$$[p_1(x, y) = x^3 - xy^2], [p_2(x, y) = x^3 + y^3].$$

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Remark. The germs $[0]$, $[x^3]$, $[x^2y]$ that correspond to the remaining normal cubic forms are also inequivalent (as germs).

Case C: $\text{corank} = 2$ and $\text{codimension} = 4$

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- 2 The residual singularity of $[f]$ is equivalent to exactly one of the following germs of two variables:

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In other words, $\text{cor}[f] = 2$ and $\text{cod}[f] = 4$ if and only if $[f]$ is equivalent to the germ of the function:

$$(x_1, \dots, x_n) \mapsto -x_1^2 - \dots - x_s^2 + x_{s+1}^2 + \dots + x_{n-2}^2 + g(x_{n-1}, x_n)$$

where $0 \leq s \leq n-2$ is the index of $[f]$ and $g(x, y)$ is one of the polynomials listed in the theorem above.

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It follows easily that the germs $[x], [y], [x^2], [y^2]$ define a basis of $m_2/\mathcal{J}[g]$ and therefore $\text{cod}[g] = 4$. □

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Therefore $[g] \stackrel{\text{equiv}}{\sim} [T_g^4]$, so we may assume that g is a polynomial of the form

$$p(x, y) + h(x, y)$$

where p is a cubic form on $\mathbb{R}^2$ and h is a homogeneous polynomial of degree 4.

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- Thus, $[g] \stackrel{\text{equiv}}{\sim} [x^2y + h(x, y)]$ where $h(x, y) = ay^4 + bxy^3 + x^2q(x, y)$ is a homogeneous polynomial of degree 4.

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- Using a sequence of smooth transformations, we can achieve the following further simplifications:

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Hint. It can be verified by direct calculation that $[-x^2y - y^4][\phi] \neq [x^2y + y^4]$ using the Taylor polynomials of $\phi = (\phi_1, \phi_2)$ for $x = 0$ and $y = 0$.

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The codimension is equal to 8 if $c \neq \pm 1$. The germs corresponding to different parameter values c and c' are equivalent if and only if $c, c' > -1$ and $c' = \frac{3-c}{1+c}$. Thus, a finite classification of germs is not possible in this case.

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